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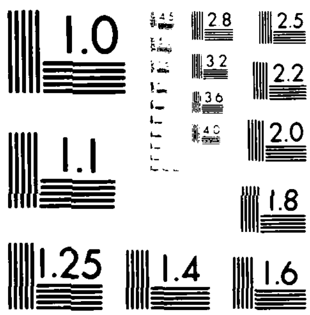

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BIRKHOFF ORBITS FOR NON-MONOTONE TWIST MAPS

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ABSTRACT

In this report we show that recent theorems of Aubry and Mather [1,6] on the existence of Birkhoff periodic orbits for monotone twist maps may be extended to a more general class of twist maps. In fact, the proofs are precisely the same as the topological proofs given in [4] once the 'correct' definitions of a Birkhoff periodic point is given for the more general family of twist maps.

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SIGNIFICANCE AND EXPLANATION

Recently it has been shown that twist maps of the annulus, which arise naturally in such diverse problems as the 3-body problem, billiards in a convex table and Hopf bifurcation of maps of the plane, possess many simple periodic orbits and quasi-periodic orbits. The original proofs of this used a variational technique requiring a globally defined 'action' and which limited the theorem to so called 'monotone' twist maps. The restriction to monotone twist maps is fairly severe since, for example, the composition of a monotone twist map with itself need not be a monotone twist map, [see 1, 6].

Topological techniques may also be used to obtain these theorems [see 4] and it is the purpose of this report to point out that, after suitably altering the definitions, the topological arguments apply to a much larger family of twist maps.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

BIRKHOFF ORBITS FOR NON-MONOTONE TWIST MAPS

Glen Richard Hall

Introduction

Recent theorems of Aubry and Mather [see 1,6] have shown that area preserving monotone twist maps of an annulus have periodic and quasi-periodic orbits of all possible rotation numbers. The techniques are "variational" requiring the existence of a generating function which is globally defined, which is a consequence of the *monotone twist condition*. Also, Bernstein [2] has given a proof replacing the graph intersection property for the area-preservation for monotone twist maps. This monotone twist condition is quite restrictive however. In particular the composition of a monotone twist map with itself need not be a monotone twist map.

In this report we show that the topological techniques of [4] may be used to extend the Aubry-Mather results to a more general family of twist maps which have been studied in Herman [5]. The major difficulty in this extension rests in the definitions and properties of the 'Birkhoff' periodic orbits. The proofs of the major theorems, once these definitions are made, are precisely the same as the topological proofs in monotone twist map case [4] and hence will only be sketched.

Twist Maps : Definition and Properties:

We let $A = \{(x,y) \in \mathbf{R}^2 : 0 \le y \le 1\}$ and $\left.\begin{matrix} p_x \\ p_y \end{matrix}\right\} : A \to \mathbf{R} : (x,y) \to \left\{\begin{matrix} x \\ y \end{matrix}\right.$ respectively. We will study diffeomorphisms $f : A \to A$ which for simplicity will be assumed to be C^∞ throughout.

Definition: If $f : A \to A$ is an orientation preserving diffeomorphism which preserves boundary components then we may define a continuous function

$$\omega_f : A \longrightarrow \mathbf{R}$$

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such that $\omega_f(x,y)$ is the angle between $D_{(x,y)}f\binom{0}{1}$ and the x-axis, where $D.f$ is the derivative matrix of f and we choose $\omega_f(x,0) \in [0,\pi]$.

Remark: The function ω_f is well defined since f is a diffeomorphism.

Definition: We say a diffeomophism $f : A \to A$ is a twist map if it satisfies

1) f preserves orientation and boundary components.

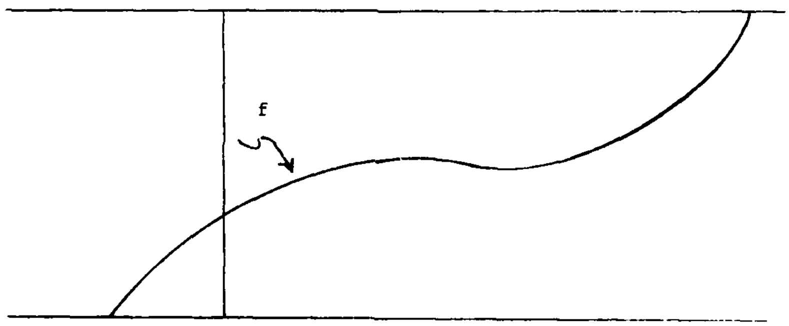
2) $\forall\ (x,y) \in A,\ \ f(x+1,y) = f(x,y) + (1,0)$, i.e., f is the lift of a diffeomorphism of an annulus,

3) $\forall\ (x,y) \in A,\ \ \omega_f(x,y) < \pi/2$.

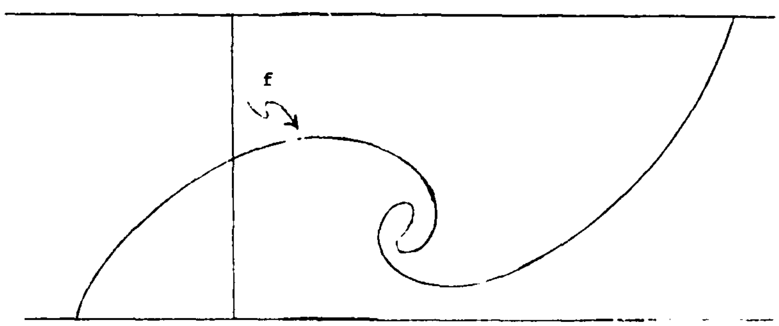
Remark: This extends the definition of monotone twist map where (3) is replaced by

$$\forall\ (x,y) \in A,\ -\pi/2 < \omega_f(x,y) < \pi/2.$$

As noted in the introduction, the composition of monotone twist maps need not be a monotone twist map, however, twist maps are closed under composition. (See Herman [5] and Fig 1).



monotone twist map



twist map

Figure 1

One of the several different ways of describing the nature of periodic orbits of a given twist map is to suspend the map to a flow and consider the nature of the associated orbits in $\mathbb{R}^3$. The need for a 'nice' suspension motivates the following:

Lemma 1: If $f : A \to A$ is a twist map then there exists a map $\psi_f : A \times [0,1] \to A$ which is a C^∞ map such that

1) for each $t \in [0,1]$, $\psi_f(\cdot,t) : A \to A$ is a twist map

2) $\psi_f(\cdot,0) = \text{identity}$, $\psi_f(\cdot,1) = f(\cdot)$.

(i.e., we may suspend f through twist maps).

Idea of the proof: This lemma is precisely the same as for monotone twist maps, i.e., we deform $f|_{\{(x,0):x\in\mathbb{R}\}}$ to the identity through circle diffeomorphisms and the vector field $D_{(x,y)}f\binom{0}{1}$, where $D.f$ is the derivative matrix for f, to the constant vector field $\binom{0}{1}$, see [4] for details. □

Next we consider how the orbits of two points may 'link' under a twist map. There are several different ways of describing this linking each with its own advantages.

Let $z_o, w_o \in A$ be fixed points of a twist map $f : A \to A$. Let $\psi_f : A \times [0,1] \to A$ be a suspension of f as in lemma (1) and let $\phi_f : A \times [0,1] \to A$ be obtained from ψ_f as follows:

$$\forall\, z \in A,\quad \phi_f(z,t) = g(h(\psi_f(z,t),t),t)$$

where $g,h : A \times [0,1] \to A$ are C^∞ and $\forall\, t \in [0,1]$, $g(\cdot,t)$, $h(\cdot,t)$ are diffeomorphisms, moreover,

1) $g(\cdot,0) = h(\cdot,0) = g(\cdot,1) = h(\cdot,1) = \text{identity}$

2) $\forall\, y_1,y_2 \in [0,1]\ \forall\, x,\ p_x(h(x,y_1)) = p_x(h(x,y_2))$

3) $\forall\, (x,y) \in A,\ p_x(g(x,y)) = x$

4) $\forall\, t \in [0,1],\ \phi_f(z_o,t) = z_o$,

i.e., by composing ψ_f with maps which preserve the x = constant foliation, we fix z_o for all time. Note that $\phi_f(\cdot,t)$ is a twist map for all t and $\phi_f(\cdot,1) = f(\cdot)$.

Next let A_z denote the universal covering space of $A \sim \{z\}$ and $\overline{\phi}_f(\cdot,t) : A_z \times [0,1] \to A_z$ the life of ϕ_f which satisfies $\overline{\phi}_f(\cdot,0) = \text{identity on } A_z$. Finally, let $\sigma : [p_y(z_o),1] \to A$ be given by $\sigma(t) = (p_x(z_o),\ t)$.

Lemma 2: The following are equivalent:

a) the braid given by $\psi_f(w_o,[0,1])$, $\psi_f(z_o,[0,1])$ is non-trivial,

b) the loop $\phi_f(w_o,[0,1])$ is non-trivial and has negative winding number about w_o,

c) if $\bar{w}_o$ is a lift of w_o to A_z then $\bar{\phi}_f(\bar{w}_o,1) \neq \bar{w}_o$,

d) the loop formed by σ followed by a segment on $\{(x,1) : x \in \mathbb{R}\}$ followed by $f \circ \sigma$ traversed in the direction of decreasing t is not contractible in $A \sim \{w_o\}$, moreover, it has negative winding number about w_o. (see Fig. 2)

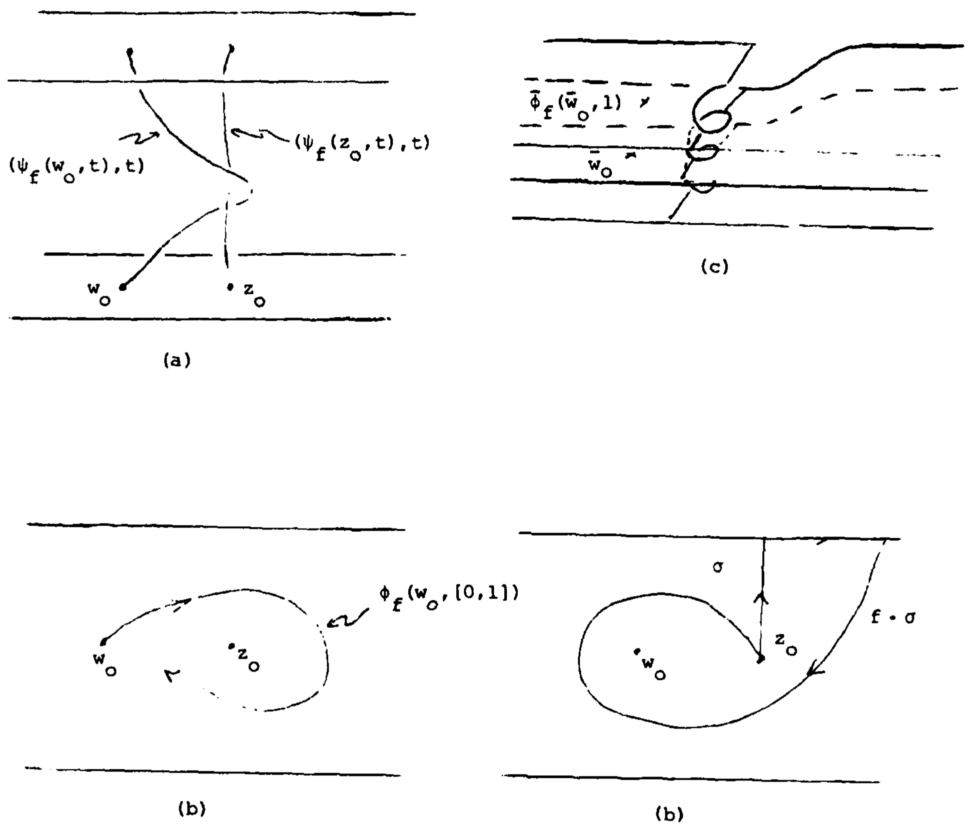


Figure 2

Proof of lemma 2: The equivalence of these statements follows easily once we have established that, for example, the loop $\phi_f(w_o,[0,1])$ has negative winding number about z_o. Suppose $p_x(w_o) < p_x(z_o)$. To show this, let

$\sigma_1 : [0,p_y(z_o)] \to A : s \to (p_x(z_o),s)$ and let $\bar{\sigma}_1$ be a lift of σ_1 to A_z.

It follows from the twist condition that for all t, $\bar{\phi}_f(\bar{\sigma}_1([0,p_y(z_o)]),t) \cap \bar{\sigma}_1([0,p_y(z_o)]) = \emptyset$ and that if $\bar{c} : [\pi/2,\infty) \to A$ is a lift of

$c : [\pi/2,\infty) \to A : \theta \to z_o + (\varepsilon \cos(-\theta), \varepsilon \sin(-\theta))$ for small $\varepsilon > 0$ with

$\bar{c}(0) \in \bar{\sigma}_1([0,p_y(z_o)])$ then $\bar{c}([\pi/2,\infty))$ contains points of $\bar{\phi}_f(\bar{\sigma}_1([0,p_y(z_o)],t)$ for all t, the proof is complete. □

Definition: We say the points z_o, w_o link under f if any of (a-d) above hold.

Remarks: 1) Note that by condition (d), none of the others depend on the choice of ψ_f or ϕ_f.

2) The conditions (a) and (c) above may also be made to include the 'direction of winding', e.g., in (a) we could say that the braids formed by twist maps require only half of the usual generaters for the braid group (i.e., 'positive braids') and in (c) if A_z is embedded in R^3 as shown in figure (2c) then $\bar{\phi}_f(\bar{w}_o,1)$ is above $\bar{w}_o$.

Lemma 3: Suppose $z_1,z_2,z_3 \in A$ are fixed points of f a twist map, and $p_x(z_1) < p_x(z_2) < p_x(z_3)$. If z_1 links with z_3 then either z_1 links with z_2 or z_3 links with z_2.

Proof of lemma 3: Suppose z_1 and z_3 do not link with z_2. Let $\phi_f(z_2,t) = z_2$ for all t. Let A_2 be the universal covering space of $A \sim \{z_2\}$, $\bar{\phi}_f$ the lift of ϕ_f satisfying $\bar{\phi}_f(\cdot,0) =$ identify on A_2 and $\bar{z}_3$ a lift of z_3. Let

$\sigma : [0,1] \to A : s \to (p_x(z_2),s)$ and $\bar{\sigma} : [0,1] \sim \{p_y(z_2)\} \to A_2$ a lift of

$\sigma : [0,1] \sim \{p_y(z_2)\} \to A$ such that the components of $\bar{\sigma}$ are connected by the lift of an arc $\theta \to z_2 + \varepsilon(\cos(-\theta), \sin(-\theta))$ where $\theta \in [-\pi/2, \pi/2]$, $\varepsilon > 0$, and so that $\bar{z}_3$ and $\bar{\sigma}\big|_{(p_y(z_2),1)}$ are also connected by such an arc.

Let $\zeta_1 : (-\infty,p_x(z_1)] \to A : s \to (s,p_y(z_1))$ and $\zeta_3 : [p_x(z_3),\infty) \to A : s \to (s,p_y(z_3))$.

Let $\bar{\zeta}_3$ be the lift of ζ_3 to A_2 containing $\bar{z}_3$. Since z_1 and z_3 link, the curve formed by $\bar{\zeta}_3$ followed by $\bar{\phi}_f(\bar{\zeta}_3([p_x(z_3),\infty)), 1)$ will surround some $\bar{z}_1$ in A_2, a lift

of z_1. Let $\bar{\zeta}_1$ be a lift of ζ_1 containing $\bar{z}_1$. By the twist condition we know that $\bar{\phi}_f(\bar{\zeta}_3([p_x(z_3),\infty)),1) \cap \bar{\sigma}((p_x(z_2),1]) = \emptyset$. But by the choice of $\bar{z}_1$ we know the curve $\bar{\zeta}_1$ followed by $\bar{\phi}_f(\bar{\zeta}_1((-\infty,p_x(z_1)]),1)$ must surround $\bar{z}_3$, i.e., we must have $\bar{\phi}_f(\bar{\zeta}_1((-\infty,p_x(z_1)]),1) \cap \bar{\sigma}([0,p_x(z_2))) \neq \emptyset$. But this contradicts the twist condition, Hence z_1 must link with z_2 and the proof is complete. □

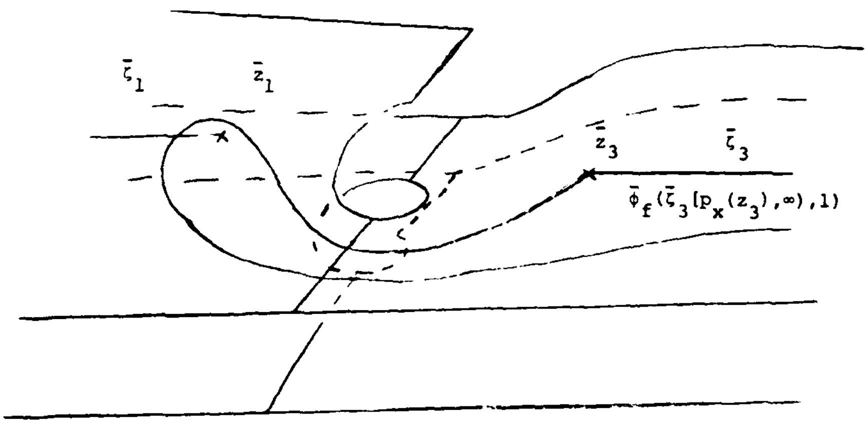


Figure 3

<u>Periodic Orbits : Definitions and Properties</u>:

Fix a twist map $f : A \to A$ and relatively prime integers p and q.

<u>Definition</u>: A point $z \in A$ is called a p/q - <u>periodic point</u> of f if

$$f^q(z) = z + (p,0) \ .$$

<u>Notation</u>: We let $g : A \to A$ denote the map given by

$$\forall\ z \in A,\ g(z) = f^q(z) - (p,0) \ ,$$

Then g is a twist map and the p/q - periodic points of f are precisely the fixed points of g. We let ψ_f and ψ_g denote suspensions of f and g respectively as described in the last section (so we could take

$$\psi_g(\cdot,t) = \psi_f(f^{[qt]}(\cdot),qt-[qt]) - t(p,0)$$

where $[\cdot]$ denotes the usual greatest integer function).

__Definition__: Let $z \in A$ be a p/q-periodic point of f. Then z is called a p/q-__Birkhoff periodic point__ if whenever $z_1, z_2 \in \{f^i(z) + (j,0) : i,j \in Z\}$ the orbits of z_1 and z_2 do __not__ link under ψ_g.

__Remark__: Using lemma (2) above we may also define p/q - Birkhoff periodic points by saying that the 'braid' formed by $\{\psi_g(w,[0,1]) : w \in \{f^i(z) + (j,0) : i,j \in Z\}\}$ is trivial. Also we could say that for each z_1, z_2 as above, ϕ_g fixing z_2, homotopic to ψ_g as in lemma (2) the loop $\phi_g(z_1,[0,1])$ is contractible in $A \sim \{z_2\}$, and so forth, (see Fig. 4).

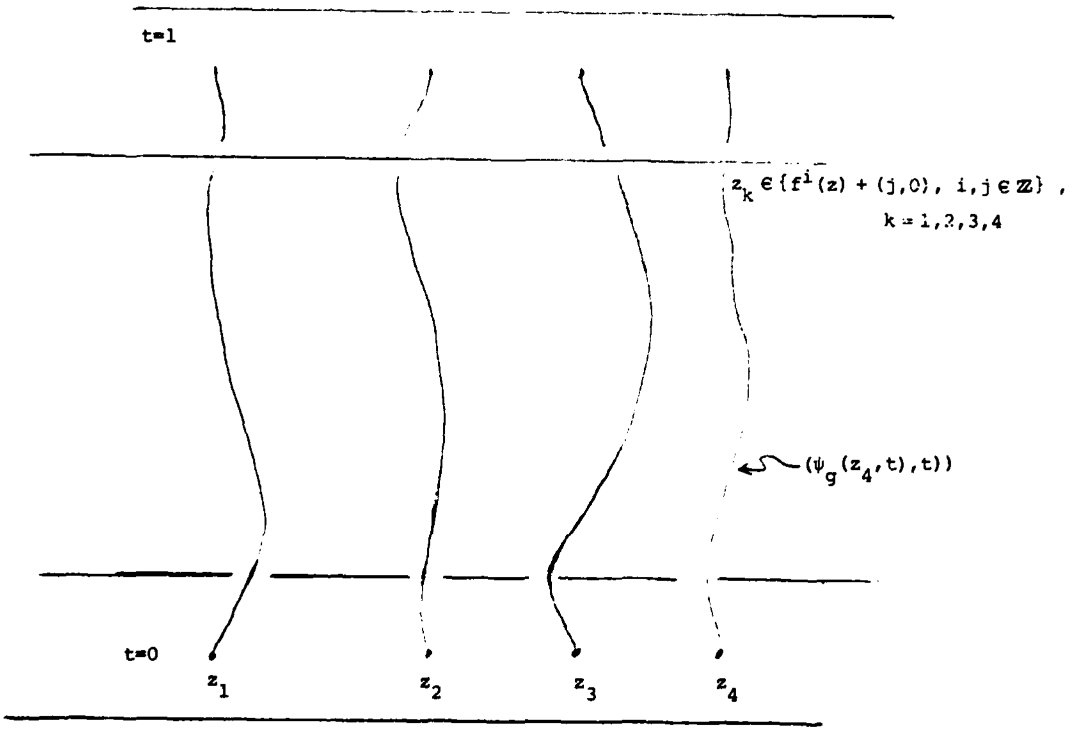
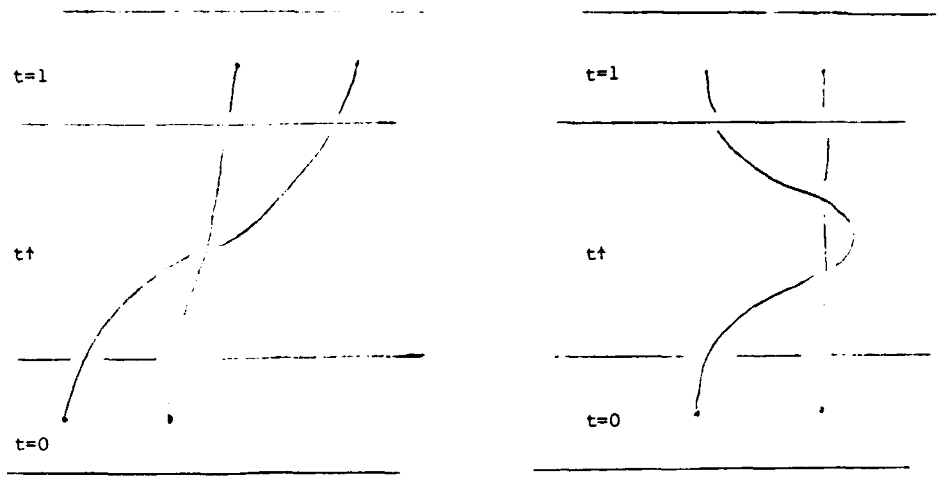


Figure 4

__Remark__: It is important to note that it is not sufficient to consider only the points $\{f^i(z) + (j,0) : i,j \in Z\}$ and f restricted to this set to determine if z is Birkhoff. For monotone twist maps the ψ_f orbits can link 'only half way' in one time-unit, however for general twist maps one point can completely encircle the other in a single unit of time (see Fig. 5).



ψ_f for monotone twist map $\qquad$ ψ_f for twist map

Figure 5.

Lemma 4: If $f_n : A \to A$, $n = 1,2, \ldots,$ is a sequence of twist maps $f_n \to f_o$ as $n \to \infty$ in the C^2 norm, $f_o : A \to A$ a twist map and if for $n = 1,2,\ldots,$ $z_n \in A$ is a p/q - Birkhoff periodic point with $z_n \to z_o$ as $n \to \infty$ then z_o is a p/q - Birkhoff periodic point for f_o.

Proof of lemma 4: Let $\psi_n : A \times [0,1] \to A$ be the suspension of f_n of the last section, $n = 0,1, \ldots$. We may assume that $\psi_n \to \psi_o$ as $n \to \infty$ in the C^1 norm.

For each $n = 0,1, \ldots$ $\forall z \in A$, $t \in [0,1]$,

$$\phi_n(z,t) = \psi_n(f_n^{[qt]}(z),qt - [qt]) - t(p,0) .$$

As above we may deform ϕ_n to a map (again called ϕ_n) which fixes z_n. If there exists $w = f_o^i(z_o) + (j,0)$ such that $\phi_o(w,[0,1])$ is not contractible in $A \sim \{z_o\}$ then for a sufficiently large n we see that $\phi_n(f_n^i(z_n) + (j,0), [0,1])$ is not contractible in $A \sim \{z_n\}$. But z_n is a p/q - Birkhoff periodic point and hence $\phi_n(f_n^j(z_n) + (j,0),[0,1])$ must be contractible in $A \sim \{z_n\}$. Since this must hold if we replace z_o with any point in $\{f_o^i(z_o) + (j,0) : i,j \in Z\}$ we see that z_o must be a p/q - Birkhoff periodic point and the proof is complete. □

Remark: Almost exactly the same argument shows that the limit of non-Birkhoff p/q - periodic points will be a non-Birkhoff p/q - periodic point.

We end this section with some useful lemmas.

Lemma 5: Suppose $z_o, w_o \in A$ are p/q - periodic points for f. With g defined as above, z_o and w_o link under g if and only if $f^i(z_o)$ and $f^i(w_o)$ link under g for all i.

Proof of lemma 5: Let $\psi_f : A \times [0,1] \to A$ be the suspension of f as in the last section. Then we may extend ψ_f to $\psi_f : A \times R \to A$ by $\psi_f(z,t) = \psi_f(f^{[t]}(z), t - [t])$. Then we note that $\psi_f(f(z_o),t) = \psi_f(z_o, t+1)$ for $t \in [0, q-1]$ and $\psi_f(f(z_o),t) = \psi_f(z_o, t-(q-1)) + (p,0)$ for $t \in [q-1, q]$. Hence if z_o, w_o link under g we see that $f(z_o)$, $f(w_o)$ must link under g since we may take $\psi_g(z,t) = \psi_f(z,qt) - t(p,0)$, (see Figure 6).

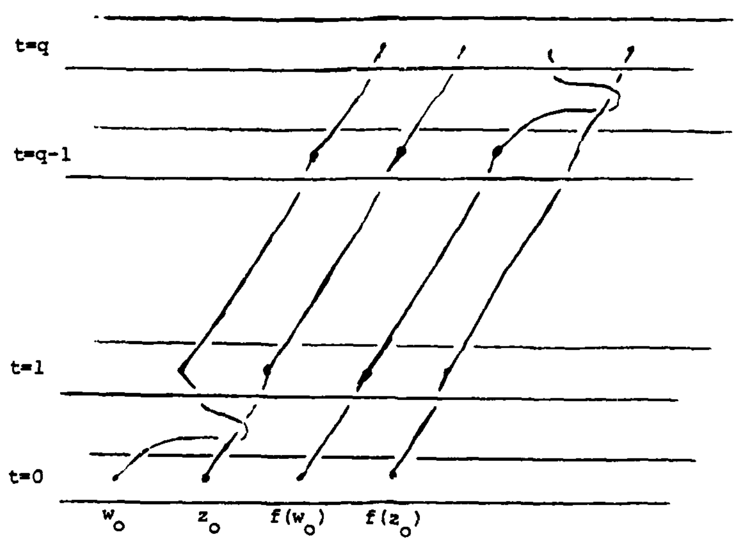


Figure 6

Similarly, if z_o, w_o link under g so must $f^{-1}(z_o)$ and $f^{-1}(w_o)$. Continuing by induction we see that z_o, w_o link under g inplies $f^i(z_o)$, $f^i(w_o)$ link under g for all $i \in Z$ and the proof is easily completed. □

<u>Lemma 6</u>: Suppose z_o is a p/q - Birkhoff periodic point and w_o is a non-Birkhoff p/q - periodic point for f. Then there exists $w_1 \in \{f^i(w_o) + (j,0) : i,j \in \mathbb{Z}\}$ such that z_o and w_1 link under g.

<u>Proof of lemma 6</u>: Let $z_o, z_1, \ldots, z_{q-1}$, denote the elements of $\{f^i(z_o) + (j,0) : i,j \in \mathbb{Z}\} \cap \{(x,y) : p_x(z_o) \leq x < p_x(z_o) + 1\}$, ordered so that $p_x(z_i) < p_x(z_{i+1})$, $i = 1,\ldots,q-1$. Similarly let $\eta_o,\ldots,\eta_{q-1}$ be the elements of $\{f^i(w_o) + (j,0) : i,j \in \mathbb{Z}\} \cap \{(x,y) : p_x(z_o) \leq x < p_x(z_o) + 1\}$ ordered so that $p_x(\eta_i) \leq p_x(\eta_{i+1})$ for $i = 1,\ldots,q-1$ and if $p_x(\eta_i) = p_x(\eta_{i+1})$ then $p_y(\eta_i) < p_y(\eta_{i+1})$. We consider two cases, either

a) for $i = 0,\ldots,q-1$, $p_x(z_i) \leq p_x(\eta_i) < p_x(z_{i+1})$, where we set $z_q = z_o + (1,0)$,

or

b) for some i, $\{(x,y) : p_x(z_i) \leq x < p_x(z_{i+1})\} \cap \{\eta_o, \ldots, \eta_{q-1}\} = \emptyset$.

In case (b) we fix i so that $\{(x,y) : p_x(z_i) \leq x < p_x(z_{i+1})\} \cap \{\eta_o, \ldots, \eta_{q-1}\} = \emptyset$ and j so that $p_x(z_j) \leq p_x(\eta_o) < p_x(z_{j+1})$. Then note that for some $k, \ell \in \mathbb{Z}$, $0 < k < q$ we have that $f^k(z_j) + (\ell,0) = z_i$. Since z_o is Birkhoff we must also have

$$f^k(z_{j+1}) + (\ell,0) = z_{i+1},$$

and hence either

$$p_x(f^k(\eta_o) + (\ell,0)) < p_x(f^k(z_j) + (\ell,0))$$

or

$$p_x(f^k(\eta_o) + (\ell,0)) \geq p_x(f^k(z_{j+1}) + (\ell,0)).$$

Hence, we see that η_o links with z_j or z_{j+1} respectively and applying lemma (5) we are done.

In case (a) we choose η_i, η_{i+1} which link under g. Since $p_x(\eta_i) < p_x(z_{i+1}) \leq p_x(\eta_{i+1})$ we may apply lemma (3) to see that either η_i or η_{i+1} links with z_{i+1} under g (note that the twist condition implies immediately that η_{i+1} and z_{i+1} link under g if $p_x(z_{i+1}) = p_x(\eta_{i+1})$). Again applying lemma (5) we see that the proof is complete. □

Corollary: If z_o is a p/q - Birkhoff periodic point for f and w_o is a p/q - periodic point for f which does not link with any point of $\{f^i(z_o) + (j,0) : i,j \in Z\}$ under g then w_o is a p/q - Birkhoff periodic point for f.

The Theorem:

In this section we outline the proof of the following theorem:

Theorem: If $f : A \to A$ is a twist map and f has a p/q - periodic point then f has a p/q - Birkhoff periodic point.

Proof of the Theorem: The proof is almost precisely the same as that given for monotone twist maps in [4]. In particular we do the following:

Step 1: Show any twist map with a p/q - periodic point is homotopic through twist maps with p/q - periodic points to a twist map with a p/q - Birkhoff periodic point.

Step 2: Show that the set of parameter values such that the homotopy has a p/q - Birkhoff periodic point is closed.

Step 3: Show the set of parameter values such that the homotopy has a p/q - Birkhoff periodic point is open.

For the remainder of this section fix relatively prime integers p, q and $f : A \to A$ a twist map with $w_o \in A$ a p/q - periodic point for f which is not Birkhoff.

Step 1: We may assume, by enlarging A and extending f if necessary, that for every $x \in R$, $f(\{x\} \times [0,1]) \cap \{x + p/q\} \times [0,1] \neq \emptyset$, (see Fig. 7). (If we can show the map f extended to the enlarged A has a p/q - Birkhoff periodic point then this point must be contained in the original strip.)

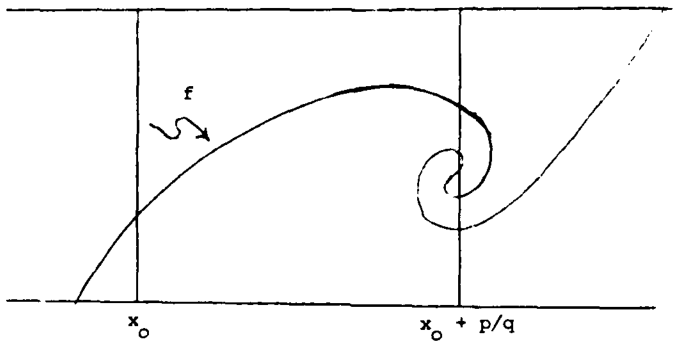


Figure 7

Choose x_o and $\delta_1 > 0$ so that $\{(x_o + ip/q+\varepsilon,y) : y \in [0,1], |\varepsilon| < \delta_1, i \in Z\}$ $\cap \{(p_x(f^i(w_o) + (j,0)) : i,j \in Z\} = 0$. Then $\exists\delta > 0$, $\delta < \delta_1$ such that $\inf\{\|f^i(w_o) + (j,0) - f((x_o + kp/q,y))\| : i,j,k \in Z, y \in [0,1]\} > \delta$ (i.e. if $f^i(w_o) + (j,0) = f(x_o + kp/q,y)$ then $f^{i-1}(w_o) + (j,o) = (x_o + kp/q,y)$ which contradicts the choise of δ_1, by compactness of the annulus the required δ exists). We form the required homotopy by composing f with maps g_s and k_s: $A \to A$, $s \in [0,1]$, with $g_o = k_o$ = identity. The maps g_s preserve the vertical foliation, the maps k_s preserve the image of the vertical foliation under f with g_s pushing points on $\{(x_o + ip/q,y) : y \in [0,1]\}$ and k_s pushing points along $\{f(x_o + ip/q,y) : y \in [0,1]\}$ up so that the map kofog, has a p/q-Birkhoff periodic point on the lines $\{(x_o + ip/q,y) : i \in Z, y \in [0,1]\}$. Moreover, the maps g_s, k_s can be required to be the identity off δ neighborhoods of the lines $\{x_o + ip/q\} \times [0,1]$ and $f(\{x_o + ip/q\} \times [0,1])$ and so each map in the homotopy is a twist map with w_o a p/q - periodic point. Let $H : A \times [0,1] \to A$ denote this homotopy of f.

<u>Step 2</u>: Let $\Xi = \{s \in [0,1] : \forall s_1 > s, H(\cdot,s_1)$ has a p/q - Birkhoff periodic point$\}$. Note $1 \in \Xi \neq \emptyset$. That Ξ is closed follows immediately from lemma (4).

Step 3: To show that Ξ is open we fix $s \in \Xi$ and let $h(\cdot) = H(\cdot,s)$. then h has w_o as a p/q - periodic point which is not Birkhoff and a p/q - Birkhoff periodic point, call it z_o. Let $z_1 \in \{h^i(z_o) + (j,0) : i,j \in \mathbb{Z}\}$ such that any $z \in \{h^i(z_o) + (j,0) : i,j \in \mathbb{Z}\}$ satisfies $p_x(z) \leq p_x(z_o)$ or $p_x(z) \geq p_x(z_1)$, i.e., z_1 is the next element of the orbit of z_o to the right of z_o. By lemma (6), with g defined as usual by

$$\forall z \in A,\ g(z) = h^q(z) - (p,0),$$

we have that there exist $\zeta_o,\dots,\zeta_r \in \{h^i(z_o) + (j,0) : i,j \in \mathbb{Z}\}$ such that $p_x(\zeta_i) < p_x(\zeta_{i+1})$ $i = 0,\dots,r-1$ and w_o links with $\zeta_o,\dots,\zeta_r$ under g (and w_o links with no other elements of $\{h^i(z_o) + (j,0) : i,j \in \mathbb{Z}\}$). We may choose $k,\ \ell \in \mathbb{Z}$ such that $h^k(\zeta_r) = z_o + (\ell,0)$, so by lemma (5), $h^k(w_o)$ links with $z_o + (\ell,0)$ under g, i.e., $h^k(w_o) - (\ell,0)$ links with z_o under g. But $h^k(w_o) - (\ell,0)$ does not link under g with z_1 since if it did we would have that w_o and $h^{-k}(z_1 + (\ell,0))$ link under g. But $p_x(z_o) < p_x(z_1)$ so $p_x(\zeta_r) = p_x(h^{-k}(z_o + (\ell,0)) < p_x(h^{-k}(z_1 + (\ell,0)))$, since z_o is a p/q - Birkhoff periodic point, and we have a contradiction of the choice of ζ_r. Hence $h^k(w_o) - (\ell,0)$ links with z_o but not with z_1 under g.

Similarly we can find k_1,ℓ_1, so that $h^{k_1}(w_o) - (\ell_1,0)$ links with z_1 and not with z_o under g (namely chose k_1,ℓ_1 so that $h^{k_1}(\zeta_0) = z_1 + (\ell_1,0)$). This is the set up for the following lemma.

Lemma 7: Suppose z_o, z_1, η_o, η_1 and $g : A \to A$ are as above. Then there exists a fixed point $\alpha \in A$ for g such that α does not link with z_o or z_1 under g and $p_x(z_o) < p_x(\alpha) < p_x(z_1)$.

Idea of the Proof of Lemma 7: The proof of this lemma is precisely the same as that of lemma (4) of [4]. We may use either braid theory and the theorem of Matsuoka [7], or consider the image under g of the set $B = \{(x,y)) \in A : p_x(z_o) \leq x \leq p_x(z_1)\}$, i.e., $g(B) \cap B$ must contain a component T which stretches across B in such a way that the map $g : g^{-1}(T) \to T$ must have a fixed point, (see Fig. 8). Moreover, by choice of T we may conclude that this fixed point does not link with z_o or z_1.

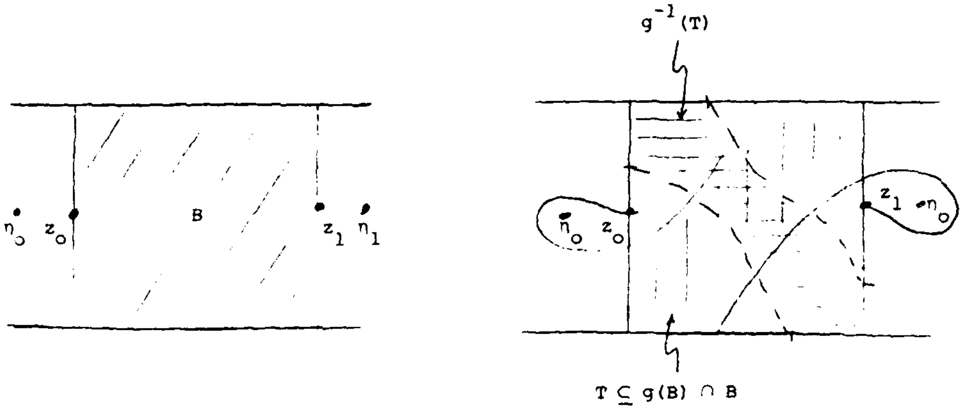


Figure 8.

As in [4], this fixed point corresponds to a p/q - *Birkhoff periodic point*. Finally, we note as in [4], for $\tilde{h} : A \to A$ a twist map sufficiently close to h, $\tilde{h}$ will give rise to $\tilde{g} : A \to A$ which (although $\tilde{g}$ may not have fixed points near η_o, η_1, z_o or z_1) maps T as shown in figure 8. Hence $\tilde{h}$ will also have a p/q - Birkhoff periodic point.

This shows that Ξ is open and hence $f : A \to A$ must have a p/q - Birkhoff periodic point, which completes the proof of the theorem 1. □

Quasi-periodic orbits:

We conclude by noting some other similarities between monotone and general twist maps.

Lemma 8: If $f : A \to A$ is a twist map, $z \in A$ a p/q - Birkhoff periodic point for f with p, q relatively prime then there exists a Lipschitz function $\rho : \mathbb{R} \to [0,1]$ such that

1) $\forall\, t,\ \rho(t + 1) = \rho(t)$,

2) $\{f^i(z) + (j,0) : i,j \in \mathbb{Z}\} \subseteq \{(t,\rho(t)) : t \in \mathbb{R}\}$.

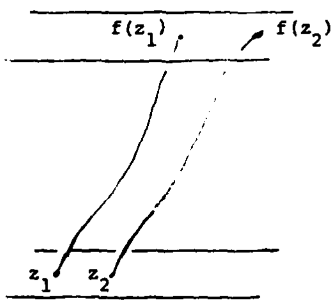
Moreover the Lipschitz constant of ρ may be chosen independent of p/q and z.

Lemma 9: If $f : A \to A$ is a twist map and for each n, f has a p_n/q_n - Birkhoff periodic point z_n where p_n/q_n converge to $\alpha \notin \mathbb{Q}$ as $n \to \infty$ and if $z_n \to w$ then $\{f^i(w) + (j,0) : i,j \in \mathbb{Z}\}$ lies on the graph of a Lipschitz curve as in lemma (8) and the

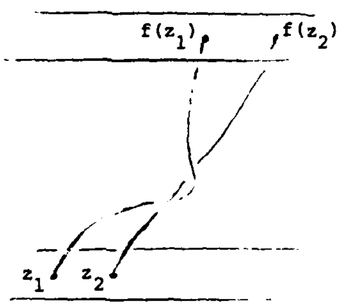
map $h : \{p_x(f^i(w) + (j,0)) : i,j \in \mathbb{Z}\} \to \{i\alpha + j : i,j \in \mathbb{Z}\}$ defined by $h(p_x(f^i(w) + (j,0))) = i\alpha + j$ is monotonic.

The proofs of these lemmas are precisely as they are for monotone twist maps. In fact if we let $f : A \to A$ be a twist map then we may define a Birkhoff point as follows:

Definition: A point $z \in A$ is a Birkhoff point for f if whenever $z_1, z_2 \in \{f^i(z) + (j,o) : i,j \in \mathbb{Z}\}$ if $p_x(z_1) = p_x(z_2)$ then $z_1 = z_2$, if $p_x(z_1) < p_x(z_2)$ then $p_x(f(z_1)) < p_x(f(z_2))$ and the arcs $t \to (\psi_f(z_1,t),t), t \to (\psi_f(z_2,t),t)$ do not link as braids in $A \times [0,1]$ (see Fig. 9).



Birkhoff



Not Birkhoff

Figure 9.

Then, as is the case with monotone twist maps, the limit of Birkhoff points is Birkhoff.

Using these results we may apply the theorem of Birkhoff on existence of periodic points for area preserving twist maps [3] with the theorem above to obtain a theorem of Aubry and Mather [1,6] for non-monotone twist maps. The details are precisely as in [4].

Concluding Remarks:

The discussion above allows the generalization of a theorem of Aubry and Mather from monotone twist maps to a more general family of twist maps. However, the twist condition described above is still not the most general which has been studied. For example, the original work of Birkhoff on twist maps required only a twist condition on the boundary of A, e.g., the upper boundary maps to the right and the lower to the left, see [3].

One might hope that a version of the above theorem holds for such a weak twist condition. We don't know the answer to this, however it is clear that substantial revisions would be needed in the arguments above. In particular the relationship between Birkhoff and non-Birkhoff periodic points (lemmas (3) and (6)) becomes much more ellusive. Perhaps the algebraic theorems such as Matsuoka [7] on the linking of orbits provide the best hope for a better theorem, but we don't know.

Finally, we have noted that a twist map $f : A \to A$ has a suspension $\psi_f : A \times [0,1] \to A$ such that $\psi_f(\cdot,t)$ is a twist map for each t. It is natural to ask if we can find a suspension so that $\psi_f(\psi_f(\cdot,s)^{-1},t)$ is a twist map for all $s < t$, i.e., so that ψ_f is twisting the vertical foliation to the right as t increases at each t. (This would make Lemma (3) trivial). We do not know if all twist maps have such "twist flows" as suspensions. (D. Bernstein has isnformed me that he can decompose any given twist map into a composition of maps with "small twist".)

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In this report we show that recent theorems of Aubry and Mather [1,6] on the existence of Birkhoff periodic orbits for monotone twist maps may be extended to a more general class of twist maps. In fact, the proofs are precisely the same as the topological proofs given in [4] once the 'correct' definitions of a Birkhoff periodic point is given for the more general family of twist maps.

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